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Eigenvalue equation of squeezed coherent states and time evolution of squeezed states

G M Saxena

Time and Frequency Section, National Physical Laboratory, Dr K S Krishnan Road, New Delhi 110012, India

E-mail: gmsaxena@csnpl.ren.nic.in

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Abstract
In this paper, we define the eigenvalue equation of the squeezed coherent state using recently introduced inverse annihilation and creation operators. The salient feature of the eigenvalue equation of the squeezed coherent state is that it consistently yields the eigenvalue equation of the squeezed vacuum when the coherent amplitude is reduced to zero. The squeezed coherent state defined by others, without using the inverse operators, cannot be reduced to the squeezed vacuum state. The time evolution of the squeezed states is also discussed. The squeezed states have phase-dependent quantum noise. The phase operators have an important role in the squeezing experiments. We show that the inverse operators are also useful in defining the phase operators. We also discuss the eigenvalue equation of the squeezed coherent state defined by Yuen *vis-à-vis* that is introduced in this paper using inverse annihilation and creation operators for bosons.

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1. Introduction

In a recent paper [1] the eigenvalue equation of the squeezed vacuum has been discussed using the inverse annihilation and creation operators. We extend the application of these inverse operators to introduce the eigenvalue equation of the squeezed coherent state and phase operator. The squeezed coherent states are useful in laser cooling and precision metrology. These states are obtained by first squeezing the vacuum and then displacing the squeezed vacuum by coherent amplitude. It is also possible to obtain the squeezed coherent states by first displacing the vacuum followed by squeezing the displaced vacuum. The choice of ordering of the displacement and squeeze operators depends on the application. We first briefly describe the properties of the inverse annihilation and creation operators. We then define the eigenvalue equation of the squeezed coherent states and discuss its properties *vis-à-vis* the squeezed coherent states introduced by Yuen [2]. We observe that in the case of a squeezed coherent state, eigenvalues are independent of the coherent amplitude and depend only on the squeeze factor.

The annihilation a and creation $a^\dagger$ operators for the bosons are defined by their action on the number state $|n\rangle$ as follows,

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{1}$$

$$a^\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle \tag{2}$$

and

$$a^\dagger a|n\rangle = n|n\rangle. \tag{3}$$

Similarly, the inverse annihilation and creation operators [1] are defined by their action on the number states as

$$a^{-1}|n\rangle = |n+1\rangle/\sqrt{(n+1)} \tag{4}$$

$$a^{\dagger^{-1}}|n\rangle = \begin{cases} |n-1\rangle/(n)^{1/2} & \text{for} \quad n = 1, 2, \ldots \\ 0 & \text{for} \quad n = 0 \end{cases} \tag{5}$$

where a^{-1} and $a^{\dagger^{-1}}$ are the right and the left inverses of a and $a^\dagger$ respectively, i.e.

$$aa^{-1} = a^{\dagger^{-1}}a^\dagger = I. \tag{6}$$

These operators also satisfy the following relation:

$$a^{-1}a = a^\dagger a^{\dagger^{-1}} = I - |0\rangle\langle 0|. \tag{7}$$

Here I is the identity operator and $|0\rangle\langle 0|$ is the projection operator on the vacuum. We see that the inverse operators a^{-1} and $a^{\dagger^{-1}}$ behave as creation and annihilation operators respectively. We describe below some relations involving the inverse operators, which are useful in our discussions:

$$D(\alpha)a^{\dagger^{-1}}D^\dagger(\alpha) = (a^\dagger - \alpha)^{-1} \tag{8}$$

$$[(a^\dagger - \alpha)^{-1}, (a^\dagger - \alpha)] = |\alpha\rangle\langle\alpha|. \tag{9}$$

Here the state $|\alpha\rangle$ is the coherent state and $D(\alpha)$ is the displacement operator, i.e.

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a). \tag{10}$$

2. Eigenvalue equations

The eigenvalue equation for the squeezed vacuum $|\sigma\rangle$ has been recently defined [1] using the inverse operators as

$$a^{\dagger^{-1}}a|\sigma\rangle = \mu|\sigma\rangle \tag{11}$$

with $\mu = (\sigma/|\sigma|)\tanh|\sigma|$ being the eigenvalue. The squeezed vacuum is obtained from the application of the squeeze operator $S(\sigma)$ on the vacuum $|0\rangle$, i.e.

$$|\sigma\rangle = S(\sigma)|0\rangle = \exp\left\{\tfrac{1}{2}\left(\sigma a^{\dagger^2} - \sigma^* a^2\right)\right\}|0\rangle. \tag{12}$$

In the above equation σ is the squeeze factor. We now introduce the eigenvalue equation of the squeezed coherent state. The squeezed coherent state is basically the displaced squeezed vacuum. In the geometrical representation of the squeezed vacuum [3], the quadrature uncertainties form an ellipse unlike the circle for the vacuum state or coherent state. In the case of the squeezed coherent states, the uncertainty circle is first squeezed, converting it into an uncertainty ellipse, and then the uncertainty ellipse is displaced from the origin by the coherent amplitude α on applying the displacement operator. The conversion of the uncertainty circle into an ellipse corresponds to phase-dependent quantum noise with one quadrature showing less and the other more than the normal vacuum noise.

The squeezed coherent state $|\alpha, \sigma\rangle$ in general is defined as

$$|\alpha, \sigma\rangle = D(\alpha)S(\sigma)|0\rangle. \tag{13}$$

The order of $S(\sigma)$ and $D(\alpha)$ may be interchanged depending on the application [4].

For introducing the eigenvalue equation for a squeezed coherent state, let us define a state $|\Omega\rangle$ as an eigenstate of the operator $a^{\dagger^{-1}}(a-\alpha)$ with eigenvalue Ω, i.e.

$$a^{\dagger^{-1}}(a-\alpha)|\Omega\rangle = \Omega|\Omega\rangle. \tag{14}$$

We shall show that the state $|\Omega\rangle$ is basically the squeezed coherent state. The choice of the eigenoperator $a^{\dagger^{-1}}(a-\alpha)$ is based on the observation that the squeezed coherent state is obtained on displacing the squeezed vacuum by coherent amplitude. Besides, on putting $\alpha = 0$ we can get back the eigenoperator for the squeezed vacuum.

We now obtain the expansion of the state $|\Omega\rangle$ in terms of the number state $|n\rangle$,

$$|\Omega\rangle = \sum_{n=0}^{\infty} C_n|n\rangle. \tag{15}$$

Here C_n are the expansion coefficients. On applying the operator $a^{\dagger^{-1}}(a-\alpha)$ from the left and using the orthonormality of the number states, we obtain from equation (15) the following recursion relation:

$$(n+1)^{1/2}\,\Omega C_n = \alpha C_{n+1} + (n+2)^{1/2}\,C_{n+2}. \tag{16}$$

In order to obtain the expansion coefficient C_n let us first consider the special case of coherent amplitude $\alpha = 0$. Equation (16) with $\alpha = 0$ is reduced to

$$(n+1)^{1/2}\,\Omega C_n = (n+2)^{1/2}\,C_{n+2}. \tag{17}$$

It is identical to the recursion relation obtained from the squeezed vacuum expansion in the number state (cf equation (3.3) of [1]). Thus we observe that for $\alpha = 0$ the state $|\Omega\rangle$ reduces to squeezed vacuum. The solution of equation (17) yields

$$C_{2n} = (1-|\Omega|^2)^{1/4}\,\{2n!\}^{1/2}\,\Omega^n/2^n(n!). \tag{18}$$

Based on the above value of C_{2n} the expansion coefficient C_n in the recursion equation (16) for $\alpha \neq 0$ may be deduced and written as

$$C_n = N(1-|\Omega|^2)^{1/4} f_n(\Omega,\alpha)\,(-\Omega)^{n/2}/(2^n n!)^{1/2} \tag{19}$$

where $f_n\,(\Omega,\alpha)$ is some function of Ω and α. N is the normalization constant. On substituting the above value of C_n in equation (16) and rearranging various terms, we obtain the following recursion relation:

$$f_{n+2} - \{2\alpha/\sqrt{(-2\Omega)}\}f_{n+1} + 2(n+1)f_n = 0. \tag{20a}$$

It is similar to the recursion relation for the Hermite polynomial $H_n(x)$ [5], i.e.

$$H_{n+2}(x) - 2xH_{n+1}(x) + 2(n+1)H_n(x) = 0 \qquad \text{with} \quad x = \alpha/(-2\Omega)^{1/2}. \tag{20b}$$

Thus the function f_n is basically the Hermite polynomial of order n and the expansion coefficient C_n may be written as

$$C_n = \frac{N(1-|\Omega|^2)^{1/4}(-\Omega/2)^{n/2}H_n[\alpha/(-2\Omega)^{1/2}]}{(n!)^{1/2}}. \tag{21}$$

The expansion of the squeezed coherent state in terms of the number states has the following form [3]:

$$|\alpha, \sigma\rangle = D(\alpha)S(\sigma)|0\rangle = (1 - |\tanh r|^2)^{1/4} \exp\left[-\tfrac{1}{2}\{|\alpha|^2 + \alpha^2(\sigma^*/|\sigma|)\,\mathrm{e}^{\mathrm{i}\vartheta}\tanh r\}\right]$$

$$\times \sum_{n=0}^{\infty} \frac{(-\mathrm{e}^{\mathrm{i}\vartheta}\tanh r)^{n/2} H_n[\alpha/\{-2\,\mathrm{e}^{\mathrm{i}\vartheta}\tanh r\}^{1/2}]}{2^{n/2}(n!)^{1/2}}|n\rangle. \tag{22}$$

If we put $\Omega = \mathrm{e}^{\mathrm{i}\vartheta}\tanh r$ in equation (22) then from equations (14) and (21) it readily follows that the eigenstate $|\Omega\rangle$ of the eigenoperator $a^{\dagger^{-1}}(a - \alpha)$ is the coherent squeezed state $|\alpha, \sigma\rangle$. For $\alpha = 0$, the state $|\Omega\rangle$ readily reduces to the squeezed vacuum $|\sigma\rangle = S(\sigma)|0\rangle$. Therefore, the coherent squeezed state given by equation (22) is the eigenstate of the eigenoperator $a^{\dagger^{-1}}(a - \alpha)$ with the eigenvalue $\mathrm{e}^{\mathrm{i}\vartheta}\tanh r$. It can also be directly verified by applying the operator $a^{\dagger^{-1}}(a - \alpha)$ to the squeezed coherent state in equation (22) that it is indeed the eigenstate of the operator $a^{\dagger^{-1}}(a - \alpha)$ with eigenvalue $\mathrm{e}^{\mathrm{i}\vartheta}\tanh r$. The normalization constant N can be readily obtained from equations (15) and (21). The squeeze factor σ is related to r and ϑ by the relation

$$\sigma = r\,\mathrm{e}^{\mathrm{i}\vartheta} \qquad r = \sigma/|\sigma|. \tag{23}$$

3. Quadrature uncertainties and regions of squeezing

The expectation values $\langle q\rangle$ and $\langle p\rangle$ and uncertainties $\langle(\Delta q)^2\rangle$ and $\langle(\Delta p)^2\rangle$ in the position (q) and momentum (p) variables in the coherent squeezed state may be readily obtained and we have the following expressions:

$$\langle q\rangle = (1/\sqrt{2})[\alpha\{\cosh\sigma + (\sigma^*/|\sigma|)\sinh\sigma\} + \alpha^*\{\cosh\sigma + (\sigma/|\sigma|)\sinh\sigma\}] \tag{24}$$

$$\langle p\rangle = (1/\sqrt{2})[\alpha\{\cosh\sigma - (\sigma^*/|\sigma|)\sinh\sigma\} - \alpha^*\{\cosh\sigma - (\sigma/|\sigma|)\sinh\sigma\}] \tag{25}$$

$$\langle(\Delta q)^2\rangle = \tfrac{1}{2} + \tanh r(\tanh r + \cos\vartheta)/(1 - \tanh r^2) \tag{26}$$

$$\langle(\Delta p)^2\rangle = \tfrac{1}{2} + \tanh r(\tanh r - \cos\vartheta)/(1 - \tanh r^2). \tag{27}$$

The position q and momentum p operators correspond to quadrature operators for a harmonic oscillator. Therefore, the above uncertainties in q and p correspond to uncertainties in two quadratures.

From the above, it follows that the expectation values of position $\langle q\rangle$ and momentum $\langle p\rangle$ operators in the squeezed coherent states do not vanish, while as shown in [1] for the squeezed vacuum, these expectation values are zero. This result follows from the fact that in the case of the squeezed vacuum, we have only phase-dependent quantum fluctuations whose average value vanishes. Whereas in the case of squeezed coherent states, we have finite amplitude of the electric field and its average is non-vanishing. The uncertainties in q and p in the coherent squeezed state are same as for the squeezed vacuum [1]. As the coherent amplitude only displaces the uncertainty ellipse of squeezed vacuum, the above results are quite justified. ϑ is the phase of the squeezed state.

We can also find the regions of squeezing for the q and p from equations (26) and (27). The squeezing in the q-quadrature occurs whenever $\cos\vartheta < -\tanh r$, while the squeezing in the p-quadrature occurs for $\cos\vartheta > \tanh r$. Further, it readily follows from these two equations that the regions of squeezing in q and p are circles of radius $\frac{1}{2}$ centred at $-\frac{1}{2}$ and $\frac{1}{2}$ respectively.

4. Dynamics of the squeezed vacuum and squeezed first excited state

The squeezed vacuum and squeezed coherent states have been extensively studied recently as their applications could be very useful in precision measurements and time and frequency metrology. The time evolution of these states could be of considerable importance in view of the exciting applications of these states. But so far hardly any effort has been made to study the time evolution of these states. We shall now discuss the time evolution of the squeezed vacuum $|\sigma\rangle \equiv |\mu, +1\rangle$ and squeezed first excited state $|\sigma, 1\rangle \equiv |\mu, -2\rangle$ (cf [1] equations (3.1) and (4.3) respectively) and find the conditions under which these states retain their squeezing characteristics for all times. Here squeezed vacuum $|\sigma\rangle \equiv |\mu, +1\rangle$ and squeezed first excited state $|\sigma, 1\rangle \equiv |\mu, -2\rangle$ are the eigenstates of the bilinear annihilation operators $a^{\dagger^{-1}}a$ and $aa^{\dagger^{-1}}$ respectively. While discussing the time evolution, we use the notation of [1], i.e. the squeezed vacuum and squeezed first excited state will be denoted by $|\mu, +1\rangle$ and $|\mu, -2\rangle$ respectively.

In the case of the coherent states, which are the eigenstates of the annihilation operators, it has been shown by Glauber [6] that the coherent states remain coherent for all times provided that the time derivative of the annihilation operator is independent of the creation operators, i.e.

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = F\{a(t), t\}. \tag{28}$$

Here F is a function of a and t. Extending this condition to the squeezed vacuum which is the eigenstate of the bilinear annihilation operator, it may be readily shown that the vacuum, which is squeezed initially, remains squeezed for all times if

$$\frac{\mathrm{d}a^{\dagger^{-1}}(t)a(t)}{\mathrm{d}t} = F\{a^{\dagger^{-1}}(t)a(t), t\}. \tag{29}$$

Similarly, the condition for the squeezed first excited state $|\mu, -2\rangle = S(\sigma)|n = 1\rangle$ to remain squeezed for all times may be expressed as

$$\frac{\mathrm{d}a(t)a^{\dagger^{-1}}(t)}{\mathrm{d}t} = F\{a(t)a^{\dagger^{-1}}(t), t\}. \tag{30}$$

The above mathematical equations state that the squeezed vacuum and squeezed first excited state will retain the squeezing properties for all times provided that the time derivatives of their respective eigenoperators are independent of the creation operators, i.e. $a^\dagger$ and a^{-1}.

The time evolution of a system is generally considered either in the Heisenberg or the Schrödinger picture. We shall now consider the time evolution of the eigenstate $|\sigma\rangle \equiv |\mu, +1\rangle$, the squeezed vacuum, in the Heisenberg and Schrödinger pictures respectively, and obtain the general conditions under which the squeezed states remain squeezed for all times.

4.1. Heisenberg picture

In the Heisenberg picture, the time evolution of the system is described in terms of the time evolution of the operators while the states are time independent. The Heisenberg equation of motion for the operator $a^{\dagger^{-1}}a$ is given by the expression

$$\frac{\mathrm{i}\mathrm{d}a^{\dagger^{-1}}(t)a(t)}{\mathrm{d}t} = \left[a^{\dagger^{-1}}(t)a(t), H(t)\right] \tag{31}$$

where $H(t)$ is the Hamiltonian of the system, and $\left[a^{\dagger^{-1}}(t)a(t), H(t)\right]$ is the commutator of the operator $a^{\dagger^{-1}}(t)a(t)$ and Hamiltonian $H(t)$. We shall obtain, using equation (31), the most general form of the Hamiltonian such that the eigenstate $|\mu, +1\rangle$ remains the eigenstate of the operator $a^{\dagger^{-1}}a$ for all times as this will ensure that initially squeezed state remains a squeezed state for all times. We shall use the notation $|\mu, +1\rangle$ for the squeezed vacuum. At time $t+\tau$, τ being a infinitesimally small time increment, we may write using equation (31) the time evolution of the operator $a^{\dagger^{-1}}(t)a(t)$ as

$$a^{\dagger^{-1}}(t+\tau)a(t+\tau) = a^{\dagger^{-1}}(t)a(t) - \mathrm{i}\tau\left[a^{\dagger^{-1}}(t)a(t), H(t)\right] + O(\tau^2) + \cdots. \quad (32)$$

Here $O(\tau^2)$ is a function of τ^2. If state $|\mu, +1\rangle$ remains squeezed at instant $t+\tau$ then it should be an eigenstate of the operator $a^{\dagger^{-1}}(t+\tau)a(t+\tau)$ at time $t+\tau$ with the eigenvalue $\mu(t+\tau)$, i.e.

$$a^{\dagger^{-1}}(t+\tau)a(t+\tau)|\mu, +1\rangle = \mu(t+\tau)|\mu, +1\rangle = \{\mu(t) + \tau(\partial\mu/\partial t) + f(\tau^2) + \cdots\}|\mu, +1\rangle. \quad (33a)$$

Using equation (32) we may rewrite equation (33*a*) as

$$\left\{a^{\dagger^{-1}}(t)a(t) - \mathrm{i}\tau\left[a^{\dagger^{-1}}(t)a(t), H(t)\right] + O(\tau^2) + \cdots\right\}|\mu, +1\rangle$$
$$= \{\mu(t) + \tau(\partial\mu/\partial\tau) + f(\tau^2) + \cdots\}|\mu, +1\rangle. \quad (33b)$$

Retaining only the first-order terms in τ, as τ is infinitesimally small, we obtain from equation (33*b*) on equating on both sides the coefficients of τ,

$$\left[a^{\dagger^{-1}}(t)a(t), H(t)\right]|\mu, +1\rangle = \mathrm{i}(\partial\mu/\partial t)|\mu, +1\rangle. \quad (34)$$

As the state $|\mu, +1\rangle$ is also an eigenstate of the operator $a^{\dagger^{-1}}(t)a(t)$, the state $|\mu, +1\rangle$ is simultaneously an eigenstate of $a^{\dagger^{-1}}(t)a(t)$ and $\left[a^{\dagger^{-1}}(t)a(t), H(t)\right]$. Therefore, the operators $a^{\dagger^{-1}}(t)a(t)$ and $\left[a^{\dagger^{-1}}(t)a(t), H(t)\right]$ should commute, i.e.

$$\left[a^{\dagger^{-1}}(t)a(t), \left[a^{\dagger^{-1}}(t)a(t), H(t)\right]\right] = 0. \quad (35)$$

The most general form of the Hamiltonian, which satisfies the commutation relation (35), is given by

$$H = A(t)a^{\dagger}(t)a(t) + B(t)\exp[\mathrm{i}n\pi a^{\dagger}(t)a(t)] + B^*(t)\exp[-\mathrm{i}n\pi a^{\dagger}(t)a(t)] + \Gamma(t) \quad (36)$$

where A, Γ are real, B is a complex number and n is an integer. The time evolution operator is $\mathrm{e}^{-\mathrm{i}Ht}$.

Similarly, it could be shown that Hamiltonian H given by equation (36) also ensures that the squeezing properties of squeezed first excited state $|\sigma, 1\rangle \equiv |\mu, -2\rangle$ are retained for all times.

4.2. Schrödinger picture

In the Schrödinger picture the operators are independent of time, and the state vectors have time dependence. The time evolution of the system is defined in terms of the time evolution of the states. For the sake of completeness, we shall very briefly discuss the time evolution of the system in the Schrödinger picture. The equation of motion of the eigenstate $|\mu, +1\rangle$ in the Schrödinger picture is given by

$$\frac{\mathrm{i}\mathrm{d}|\mu, +1\rangle}{\mathrm{d}t} = H|\mu, +1\rangle. \quad (37)$$

Here $|\mu, +1\rangle$ is the eigenstate of the operator $a^{\dagger^{-1}}a$ with the eigenvalue $\mu(t)$ at the time t. We may write, using equation (37) at instant $t+\tau$ for a infinitesimal time increment τ, that

$$|\mu(t+\tau), +1\rangle = |\mu(t), +1\rangle - \mathrm{i}\tau H|\mu, +1\rangle + O(\tau^2)|\mu(t), +1\rangle + \cdots. \quad (38)$$

If $|\mu(t+\tau), +1\rangle$ is still an eigenstate of the operator $a^{\dagger^{-1}}a$ with the eigenvalue $\mu(t+\tau)$ then

$$a^{\dagger^{-1}}a|\mu(t+\tau), +1\rangle = \{\mu + \tau(\partial\mu/\partial t) + f(\tau^2) + \cdots\}|\mu(t+\tau), +1\rangle. \tag{39}$$

We obtain from equations (38) and (39)

$$\left[a^{\dagger^{-1}}a, H\right]|\mu(t), +1\rangle = \mathrm{i}(\partial\mu/\partial t)|\mu(t), +1\rangle. \tag{40}$$

Equation (40) is similar to equation (34). Therefore following the reasoning given in the Heisenberg picture, the most general form of the Hamiltonian, consistent with the requirement that the state $|\mu(t), +1\rangle$ is the eigenstate of $a^{\dagger^{-1}}a$ for all times, is given by

$$H = A(t)a^{\dagger}(t)a(t) + B(t)\exp[\mathrm{i}n\pi a^{\dagger}(t)a(t)] + B^*(t)\exp[-\mathrm{i}n\pi a^{\dagger}(t)a(t)] + \Gamma(t) \tag{41}$$

where A, Γ are real, B is a complex number and n is an integer. The Hamiltonian given either by equation (36) or (41) is consistent with the condition that squeezed vacuum retains its squeezing properties for all times.

5. Phase operators

The phase operators are useful in the applications of the coherent and the squeezed states. The phase operators introduced by Susskind–Glogower [7] are one-sided unitary. As the inverse of the annihilation and creation operators is one-sided unitary, we discuss whether the inverse of annihilation and creation operators can be used to define the phase operators. We first consider the exponential Susskind–Glogower phase operators defined as [8]

$$\mathbf{e}_{\mathrm{s}}{}^{\mathrm{i}\varphi} = \sum_{n=0}^{\infty}|n\rangle\langle n+1| \tag{42}$$

$$\mathbf{e}_{\mathrm{s}}{}^{-\mathrm{i}\varphi} = \sum_{n=0}^{\infty}|n+1\rangle\langle n|. \tag{43}$$

These operators satisfy the condition that

$$\mathbf{e}_{\mathrm{s}}{}^{\mathrm{i}\varphi}|n\rangle = |n-1\rangle \tag{44}$$

$$\mathbf{e}_{\mathrm{s}}{}^{-\mathrm{i}\varphi}|n\rangle = |n+1\rangle \tag{45}$$

and

$$\mathbf{e}_{\mathrm{s}}{}^{\mathrm{i}\varphi}|0\rangle = 0. \tag{46}$$

We express these phase operators in terms of the inverse annihilation operator and obtain the following relations:

$$\mathbf{e}_{\mathrm{s}}{}^{\mathrm{i}\varphi} = aN^{-1/2} \quad \text{and} \quad \mathbf{e}_{\mathrm{s}}{}^{-\mathrm{i}\varphi} = N^{1/2}a^{-1}. \tag{47}$$

We observe that

$$\mathbf{e}_{\mathrm{s}}{}^{\mathrm{i}\varphi}\mathbf{e}_{\mathrm{s}}{}^{-\mathrm{i}\varphi} = aN^{-1/2}N^{1/2}a^{-1} = I \tag{48a}$$

as $aa^{-1} = I$.

Similarly,

$$\mathbf{e}_{\mathrm{s}}{}^{-\mathrm{i}\varphi}\mathbf{e}_{\mathrm{s}}{}^{\mathrm{i}\varphi} = N^{1/2}a^{-1}aN^{-1/2} = I - |0\rangle\langle 0|. \tag{48b}$$

The above follows from the properties of annihilation and its inverse operator,

$$a^{-1}a = I - |0\rangle\langle 0|.$$

Further we have

$$\mathbf{e}_{\rm s}{}^{{\rm i}\varphi}|0\rangle = aN^{-1/2}|0\rangle = 0. \qquad (48c)$$

Thus we see that the one-sided unitary nature of a^{-1} and a is useful in defining the Susskind–Glogower phase operators. The definition of the Susskind–Glogower phase operators in terms of the annihilation and inverse of the annihilation operators satisfies the properties of the phase operators (equations (48)). The annihilation and the inverse annihilation operators are the natural choice for defining the phase operator due to the one-sided unitary property of these operators. We see that the inverse of the annihilation operator is not only useful in defining the eigenoperator for the squeezed vacuum and coherent squeezed states but it also defines the phase operator; which is a very important parameter in the studies of the squeezed states and their applications in quantum optics experiments.

6. Discussions

To highlight the salient features of the squeezed coherent state introduced in this paper let us first discuss the eigenvalue equation, defined by Yuen [2, 3], for the squeezed coherent states, $|\alpha, \sigma\rangle$, i.e.

$$\{a \cosh r - a^\dagger\, {\rm e}^{{\rm i}\vartheta} \sinh r\}|\alpha, \sigma\rangle = \{\alpha \cosh r - \alpha^*\, {\rm e}^{{\rm i}\vartheta} \sinh r\}|\alpha, \sigma\rangle. \qquad (49)$$

On putting $\alpha = 0$ in equation (49), the eigenvalue equation becomes

$$(a \cosh r - a^\dagger\, {\rm e}^{{\rm i}\vartheta} \sinh r)|\sigma\rangle = 0. \qquad (50)$$

As $\alpha = 0$ corresponds to the squeezed vacuum, equation (50) may be interpreted as the eigenvalue equation of the squeezed vacuum $|\sigma\rangle$ for eigenoperator $\{a \cosh r - a^\dagger\, {\rm e}^{{\rm i}\vartheta} \sinh r\}$ with eigenvalue zero. Yuen's eigenvalue equation (49) of the squeezed coherent state is basically the unitary equivalent of the eigenvalue equation of the coherent state. With the coherent amplitude $\alpha = 0$, the eigenvalue equation becomes true only for the eigenvalue zero, a trivial case. Whereas for $\alpha = 0$ and for finite values of squeeze factor σ, it should also be valid for eigenvalues other than zero [9]. Now let us consider the squeezed coherent state defined using the inverse annihilation operators. For the coherent amplitude $\alpha = 0$ in equation (14), the eigenvalue equation of the squeezed coherent state consistently reduces to the eigenvalue equation of the squeezed vacuum with the eigenvalues $\Omega = (\sigma/|\sigma|) \tanh|\sigma|$, a non-trivial case. This is a very distinctive feature of the eigenvalue equation of the squeezed coherent state, defined in this paper, using inverse annihilation and creation operators for bosons.

References

[1] Mehta C L, Roy A K and Saxena G M 1992 *Phys. Rev.* A **46** 1565
[2] Yuen H P 1976 *Phys. Rev.* A **13** 2226
[3] Loudon R and Knight P L 1987 *J. Mod. Op.* **34** 709
[4] Fisher R A, Nieto M M and Sandberg V D 1984 *Phys. Rev.* D **29** 1107
[5] Abramowitz M and Stegun I (ed) 1965 *Handbook of Mathematical Functions* (New York: Dover) p 300
[6] Glauber R J 1966 *Phys. Lett.* **21** 650
[7] Susskind L and Glogower J 1964 *Physics* **1** 49
[8] Bernett S M and Pegg D T 1986 *J. Phys. A: Math. Gen.* **19** 3849
[9] Saxena G M, Chatterjee A and Mathur B S 1997 Eigenvalue problem of squeezed coherent states for bosons and para-bosons (abstract book) *12th Int. Congress on Mathematical Physics (Brisbane, Australia, July 13–19)*